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Sound generation by hydrodynamic sources near a cavitated line vortex

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The paper examines the scattering properties of a cavitated line vortex when excited by line and point sources of sound. It is found that the vortex resonances discovered by Kelvin for incompressible flow are essentially unmodified by fluid compressibility, and that many of the resonant modes radiate energy to infinity. Those resonant modes dominate the vortex response, their amplitude growing algebraically with time in a driven instability of the model flow.

The off resonance response is dependent on the value of a normalized frequency parameter $(\omega/\Omega)^2 |\ln ka|$. Ω denotes the angular velocity of the steady vortex flow, k the acoustic wave-number, and a the cavity radius. Even off resonance the cavity is an extremely efficient wave scatterer, the scattering efficiency increasing with source order. For example, the scattered energy of a point quadrupole is shown to exceed that of the direct field by a factor of 10^8 for the typical underwater flow Mach number of 10^{-2}.

Introduction

The object of this paper is to consider the scattering properties of a cavitated line vortex which is excited by external line and point multipole sources of sound. Previous studies (Ffowcs Williams 1969; Ffowcs Williams & Hunter 1970) of the scattering by bodies and cavities, in particular by resonant cavities, indicate that the acoustic output of a source can be substantially increased by the presence of nearby scattering centres. The effect is more pronounced for multipole sources. This is confirmed in what follows.

The equilibrium vortex flow consists of a steady velocity $U_i(\mathbf{x})$, which has components $(0, \gamma/r, 0)$ $(r > a > 0)$ in cylindrical polar co-ordinates. At the cavity wall, $r = a$, the inertial pressure balances the hydrostatic pressure. In the unsteady state, a small irrotational velocity, $u'_i(\mathbf{x}, t) = \nabla_i \Phi$, is superposed on the mean flow. It is then required that the material derivative of the pressure should vanish on the cavity wall, the cavity being a constant pressure zone. A solution is now sought where the perturbation satisfies a radiation condition in the form of outgoing waves at infinity.

The flow field is characterized by two parameters: the Mach number, $M = U/c$ $(U = \gamma/a)$, and a Strouhal frequency S, which measures frequencies on the natural frequency scale $\Omega = U/a$. The equations of motion for a periodic perturbation at frequency $\omega = kc$ can be linearized to give Helmholtz's equation $(\nabla^2 + k^2)\,\Phi = 0$

if $M^2S = Mka \ll 1$, and $M \ll 1$. This scattering equation is solved in cylindrical polar co-ordinates for various source configurations. In doing this, we confirm Kelvin's (1910) deduction that, at low wave-numbers, $ka \ll 1$, the cavity has no axially symmetric resonance, but has resonances at frequencies $\omega_m = (m \pm |m|^{\frac{1}{2}})\,\Omega$, $(m \neq 0)$, for asymetric modes with aximuthal period $2\pi/m$. The angular phase velocity of the resonant perturbation field is always with the mean flow. We find it surprising that Kelvin's conclusions are unmodified by fluid compressibility. The resonant modes are undamped by the radiation loss, so that the mean flow must be giving up energy to maintain the resonance, and this effect is unnoticed on a linear theory. In this respect, those free waves are similar to Miles's (1958) radiating neutral waves on a supersonic shear layer, which gives rise to an infinite acoustic scattering cross-section. The vortex resonances give rise to similar infinities in the acoustic field, and would no doubt lead to breakdown of the primary flow.

We confirm this idea by studying an initial value problem where the vortex is disturbed from rest by a point impulse. The resonant modes are excited, and survive indefinitely, even though energy is constantly being lost to their acoustic field. The response of a vortex to a periodic disturbance switched on at time $t = 0$ is shown to be a superposition of resonant modes, each with amplitude increasing algebraically with time. The driven instability is therefore not violent, but must lead inevitably to a breakdown of the model flow.

The resonant response is the primary feature of the flow, but very large scattered fields can be excited at non-resonant conditions. These cases are worked out in detail for point multipole excitation of the vortex flow.

The two-dimensional field scattered by a line vortex

The inviscid fluid flow equations may be written:

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{\partial u_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{Du_i}{Dt}+\frac{c^2}{\rho}\frac{\partial \rho}{\partial x_i} = 0, \tag{2}$$

where it is assumed that disturbances are of small amplitude, so that ∇p can be approximated by $c^2\nabla\rho$, c being the assumed constant speed of sound. The symbol D/Dt denotes the material derivative.

Equations (1) and (2) can alternatively be written

$$\frac{D}{Dt}\ln\rho+\frac{\partial u_i}{\partial x_i} = 0, \tag{3}$$

$$\frac{Du_i}{Dt}+c^2\frac{\partial}{\partial x_i}\ln\rho = 0. \tag{4}$$

These equations can be combined, by way of cross-differentiation, to yield an equation which is independent of the fluid density ρ:

$$\frac{\partial^2 u_i}{\partial t^2}-c^2\nabla u_i = -\frac{\partial}{\partial x_i}\frac{D}{Dt}(\tfrac{1}{2}u^2)-c^2\frac{\partial}{\partial x_j}\left\{\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right\}-\frac{\partial}{\partial t}\left(u_j\frac{\partial u_i}{\partial x_j}\right). \tag{5}$$

We consider a small perturbation velocity $u_i'(\mathbf{x}, t)$, given by the gradient of a potential function $\Phi(\mathbf{x}, t)$, superposed on the mean flow

$$u_i(x_j, t) = U_i(x_j) + u_i'(x_j, t). \tag{6}$$

Equation (5) then yields an inhomogeneous wave equation for the velocity perturbation, which reduces to the homogeneous wave equation, provided the Mach number and Strouhal number of the flow satisfy

$$M^2 S \ll 1, \quad M \ll 1. \tag{7}$$

Under those conditions (5) becomes

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \frac{\partial^2 \Phi}{\partial x_i^2} = 0, \tag{8}$$

or, if Φ is supposed time periodic of period $2\pi/\omega$,

$$\nabla^2 \Phi + k^2 \Phi = 0 \quad (k = \omega/c). \tag{9}$$

In a cylindrical co-ordinate system, (9) has the form,

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} + k^2 \Phi = 0. \tag{10}$$

For a two-dimensional flow, which is cylindrically symmetric, (10) can be solved to give

$$\Phi(r, t) = [AH_0^{(1)}(kr) + BH_0^{(2)}(kr)] \exp(\pm i\omega t). \tag{11}$$

The radiation condition at infinity requires that $\Phi(r, t)$ be given by either $AH_0^{(1)}(kr) \exp(-i\omega t)$ or $BH_0^{(2)}(kr) \exp(i\omega t)$. We choose the former, so that

$$\Phi(r, t) = AH_0^{(1)}(kr) \exp(-i\omega t). \tag{12}$$

Consider a constant density steady potential vortex flow, for which Bernoulli's equation is

$$p/\rho_0 + \tfrac{1}{2}\gamma^2/r^2 = p_0/\rho_0, \tag{13}$$

γ being the vortex strength, and p_0 the pressure at infinity.

If the vortex is cavitated with $p = 0$ on the cavity wall, $r = a$, then

$$\tfrac{1}{2}\gamma^2/a^2 = p_0/\rho_0. \tag{14}$$

Now consider small unsteady motion about the steady state, forced by an unsteady ambient pressure. The unsteady Bernoulli equation now describes the flow,

$$\frac{\partial \Phi}{\partial t} + \int \frac{dp}{\rho} + \frac{1}{2}\frac{\gamma^2}{r^2} = \frac{p_0}{\rho} + \frac{\overline{p}}{\rho} \exp(-i\omega t), \tag{15}$$

where $p_0 + \overline{p} \exp(-i\omega t)$ is the pressure at infinity and $\rho = \rho_0 + \rho'$.

The linearized form of the Bernoulli equation is

$$\rho_0 \frac{\partial \Phi}{\partial t} + p + \tfrac{1}{2}\rho_0 \frac{\gamma^2}{r^2} = p_0 + \overline{p} \exp(-i\omega t). \tag{16}$$

The boundary condition on the cavity wall requires that

$$\frac{\partial p}{\partial t} + \frac{\partial \Phi}{\partial r}\frac{\partial p}{\partial r} = 0, \quad \text{on} \quad r = a. \tag{17}$$

Equation (13) can be used to specify $\partial p/\partial r$, so that

$$-i\omega\overline{p}+A\omega^2\rho_0 H_0(ka)+\rho_0(\gamma^2/a^3)AkH_0'(ka) = 0. \tag{18}$$

The equation for the eigenfrequencies is thus

$$\frac{H_0'(ka)}{H_0(ka)} = -\frac{c^2\rho_0}{2p_0}ka. \tag{19}$$

If the cavity radius is much less than a wavelength, it is appropriate to consider the asymptotic form of the Hankel function for small argument:

$$H_0(ka) \sim \frac{2i}{\pi}\ln ka \quad (ka \ll 1). \tag{20}$$

Equation (19) then yields

$$\ln ka = -\frac{2p_0}{c^2\rho_0}\left(\frac{1}{ka}\right)^2 = -\left(\frac{M}{ka}\right)^2. \tag{21}$$

This equation has only the trivial solution $ka = 0$, so that the present system restricted by the conditions (7) has no non-zero natural frequency, and the cavity has no axially symmetric resonant mode.

The amplitude of the scattered field is given by (18) as

$$A = \frac{\pi}{2}\frac{\omega\overline{p}}{\rho_0}\frac{1}{\Omega^2}\left[\frac{1}{(\omega/\Omega)^2\ln ka+1}\right] \quad (ka \ll 1), \tag{22}$$

a result depending on the magnitude of the non-dimensional parameter

$$(\omega/\Omega)^2\ln ka,$$

the significance of which may be demonstrated as follows.

If the cavity is considered to exist in purely static fluid, with no vortex to maintain it, i.e. if it is merely a cylindrical bubble, then in order to maintain a symmetric radial velocity v_r at the bubble boundary, a pressure p must be maintained there. The impedance of the cavity as a bubble, z_1, is the ratio p/v_r, given by the solution to the exterior compressible problem as

$$z_1 = p/v_r = \left.\frac{\rho_0 i\omega\Phi}{\partial\Phi/\partial r}\right|_{r=a} = ia\omega\rho_0\ln ka. \tag{23}$$

On the other hand, if the fluid inertia and compressibility do not control the motion, the cavity is more aptly considered as a steady incompressible vortex field changing its equilibrium radius in response to a slowly varying external pressure. The ratio of pressure to normal velocity at the cavity boundary in this case is again an impedance, whose value z_2 can be computed from the equilibrium equation,

$$\left.\begin{aligned} \frac{1}{2}\frac{\gamma^2}{a^2} &= \frac{p_0}{\rho_0}, \\ v_r &= \frac{\partial a}{\partial t} = -iw\Delta a, \\ -\frac{\gamma^2}{a^3}\Delta a &= \frac{\Delta p}{\rho_0} = \frac{p'}{\rho_0} = \frac{\gamma^2 v_r}{i\omega a^3}, \end{aligned}\right\} \tag{24}$$

and
$$z_2 = \frac{p'}{v_r} = \frac{\rho_0 \gamma^2}{i\omega a^3}. \tag{25}$$

The ratio of the two impedances then indicates which effect is dominant in determining the cavity response to an external field. The ratio $|z_1/z_2|$ is the parameter $(\omega/\Omega)^2 |\ln ka|$, which appears in (22). For low values of this parameter, the flow is controlled by the steady vortex, while at high values it is controlled by the inertia and compressibility of the external flow, in precisely the same manner as is a passive cylindrical bubble.

When inertial effects are negligible, i.e. $(\omega/\Omega)^2 |\ln ka| \ll 1$, then the potential Φ is given in the following form,

$$\Phi(r) = \frac{\pi}{2} \frac{\omega \overline{p}}{\rho_0} \frac{a^4}{\gamma^2} H_0(kr) \exp(-i\omega t), \tag{26}$$

while in the other limit, $(\omega/\Phi)^2 |\ln ka| \gg 1$,

$$\Phi(r) = \frac{\pi}{2} \frac{\omega \overline{p}}{\rho_0} \frac{1}{\omega^2 \ln ka} H_0(kr) \exp(-i\omega t). \tag{27}$$

The scattered pressure fields in these two cases are

$$p(r) = \frac{\pi i}{2} S^2 \overline{p} \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \exp[i(kr - \pi/4)] \left(kr \to \infty, \quad \left(\frac{\omega}{\Omega}\right)^2 |\ln ka| \ll 1\right), \tag{28}$$

and

$$p(r) = \frac{\pi i}{2} \frac{1}{\ln ka} \overline{p} \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \exp[i(kr - \pi/4)] \left(kr \to \infty, \quad \left(\frac{\omega}{\Omega}\right)^2 |\ln ka| \gg 1\right), \tag{29}$$

$$ka = \frac{\omega a}{c} = \frac{\omega a}{U} . \frac{U}{c} = MS.$$

If $\overline{p}$, the amplitude of the unsteady pressure field that drives the motion, is taken to be that due to a source of strength $Q/4i$, distance ξ away from the cavity centre, then

$$\overline{p} = (\omega \rho_0 / 4) Q H_0(k\xi). \tag{30}$$

It is seen that for a wavelength large compared to the radius of the cavity, i.e. $ka \ll 1$, the scattered field is of order $S^2 |\ln ka|$ times the direct field if

$$(\omega/\Omega)^2 |\ln ka| \ll 1,$$

whereas the scattered and direct fields are of the same order when

$$(\omega/\Omega)^2 |\ln ka| \gg 1,$$

provided the distance of the source from the cavity centre is of the same order as the radius of the cavity.

Now we consider an incident field due to a line dipole.

The pressure field in this case is given as follows; for large kr,

$$p(r) = iD_k x_k \left(\frac{2k}{\pi}\right)^{\frac{1}{2}} \frac{1}{r^{\frac{3}{2}}} \exp[i(kr - \pi/4)], \tag{31}$$

and the amplitude of the dipole near field, $\bar{p}$, is

$$\bar{p} = D_k \frac{\partial}{\partial x_k} H_0(kr)\bigg|_{r=\xi}, \tag{32}$$

which becomes, for small values of $k\xi$,

$$\bar{p} = \frac{2i}{\pi} D_k \frac{\xi_k}{\xi^2}. \tag{33}$$

We find that for $kr \to \infty$, the amplitude of the scattered field is of order S^2/ka, or $1/(ka \ln ka)$ times the direct field, depending on whether $(\omega/\Omega)^2 |\ln ka| \ll 1$, or $\gg 1$, respectively, provided again that the distance ξ is of the same order of magnitude as the cavity radius.

The scattering of a quadrupole field is found in a similar way. The far pressure field is given as

$$p(r) = T_{ij} x_i x_j k \left(\frac{2k}{\pi}\right)^{\frac{1}{2}} \frac{1}{r^{\frac{5}{2}}} \exp\left[i(kr - \pi/4)\right] \quad (kr \to \infty). \tag{34}$$

The amplitude of the quadrupole near field, $\bar{p}$, is, in this case,

$$\bar{p} = T_{ij} \frac{\partial^2}{\partial x_i \partial x_j} H_0(kr)\bigg|_{r=\xi}. \tag{35}$$

For the large wavelength limit, $k\xi \ll 1$, this becomes

$$\bar{p} = -\frac{2i}{\pi} T_{ij} \left[\frac{2\xi_i \xi_j}{\xi^4} - \frac{\delta_{ij}}{\xi^2}\right]. \tag{36}$$

From (36) it can be seen that the amplitude of the scattered field is of order

$$\frac{S^2}{(ka)^2}, \quad \text{or} \quad \left(\frac{1}{ka}\right)^2 \left|\frac{1}{\ln ka}\right|$$

times the direct field when $(\omega/\Omega)^2 |\ln ka|$ is respectively small or large compared to unity and $\xi \sim a$.

To give an indication of the magnitude of this effect consider the limit $(\omega/\Omega)^2 |\ln ka| \ll 1$. Since $MS = ka$, this limit implies small values of Strouhal number compared to unity. Then, for a Mach number 10^{-2}, the scattered field exceeds the direct field in mean square pressure by a factor in excess of 10^8.

Three-dimensional scattering of point source fields

Because cavities exist near inhomogeneous source systems, where the source is essentially three dimensional, and since axial dependence is naturally expected, it is more realistic to examine the scattering properties of the cavity when the incident field is that due to a point source, dipole, or quadrupole. A solution is desired in the case of rapid variation of the field on the scale of the cavity, so that it is necessary to find an exact solution to the three-dimensional problem. It is assumed here that the flow is restricted by conditions (7), so that, as before, the relevant equation of motion is (9).

Equation (9) assumes the following form in cylindrical polar co-ordinates:

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}+k^2\Phi = 0. \tag{37}$$

Solutions to this equation are well known, and, if multivalued solutions, and solutions with linear growth are excluded, then $\Phi(r,\phi,z)$ can be written as

$$\Phi(r,\phi,z) = \int_{-\infty}^{\infty}\sum_{-\infty}^{\infty} B_m(\alpha)\,H_m^{(1)}(\beta r)\exp(im\phi)\exp(i\alpha z)\exp(-i\omega t)\,d\alpha, \tag{38}$$

$B_m(\alpha)$ being the amplitude of each mode, and where

$$\begin{aligned}\beta &= +(k^2-\alpha^2)^{\frac{1}{2}} \quad (0<\alpha<k)\\ &= +i(\alpha^2-k^2)^{\frac{1}{2}} \quad (k<\alpha<\infty).\end{aligned} \tag{39}$$

The first kind of Hankel functions have been chosen in (38), in accordance with the radiation condition, and in what follows the superscript will be omitted, it being understood that the symbol $H_m(\beta r)$ means an mth-order Hankel function of the first kind.

The field of a point source situated at $(x_0, y_0, 0)$ is given by

$$\Phi(r,\phi,z) = \exp(ikR)/R, \tag{40}$$

where

$$R^2 = (x-x_0)^2+(y-y_0)^2+z^2. \tag{41}$$

It is appropriate to consider the Fourier transform with respect to z of the function Φ,

$$\bar{\phi}(r,\phi,\alpha) = \int_{-\infty}^{\infty}\frac{\exp\{ik[(x-x_0)^2+(y-y_0)^2+z^2]^{\frac{1}{2}}\}}{[(x-x_0)^2+(y-y_0)^2+z^2]^{\frac{1}{2}}}\exp(-i\alpha z)\,dz, \tag{42}$$

$$\bar{\phi}(r,\phi,\alpha) = \pi i H_0(\beta a) \tag{43}$$

(Erdelyi, Magnus & Oberhettinger 1954), where

$$a^2 = (x-x_0)^2+(y-y_0)^2 = |r-r_0|^2. \tag{44}$$

The zeroth-order Hankel function $H_0(\beta a)$ can now be expanded (Morse & Feshbach 1953) to give

$$H_0(\beta a) = \sum_{-\infty}^{\infty}\begin{Bmatrix}J_m(\beta r)\,H_m(\beta r_0)\\ H_m(\beta r)\,J_m(\beta r_0)\end{Bmatrix}\exp[im(\phi-\phi_0)]\begin{Bmatrix}(r<r_0)\\(r>r_0)\end{Bmatrix} \tag{45}$$

which enables one to write $\Phi(r,\phi,z)$ as

$$\Phi(r,\phi,z) = \frac{i}{2}\int_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\begin{Bmatrix}J_m(\beta r)\,H_m(\beta r_0)\\ H_m(\beta r)\,J_m(\beta r_0)\end{Bmatrix}\exp[im(\phi-\phi_0)]\exp(-i\alpha z)\,d\alpha\begin{Bmatrix}(r<r_0)\\(r>r_0)\end{Bmatrix}. \tag{46}$$

When the source has arbitrary strength $-2iQ$, the field can be written as

$$\Phi(r,\phi,z) = Q\int_{-\infty}^{\infty}\begin{Bmatrix}\sum_{-\infty}^{\infty} J_m(\beta r)\,H_m(\beta r_0)\\ H_m(\beta r)\,J_m(\beta r_0)\end{Bmatrix}\exp[im(\phi-\phi_0)]\exp(-i\alpha z)\,d\alpha. \tag{47}$$

Equation (47) expresses the field of the point source in terms of the natural functions of the cavity, thus enabling the matching of both fields on the cavity wall to be performed easily.

The linearized Bernoulli equation appropriate to the three-dimensional problem, where axial symmetry is no longer assumed, is

$$\rho_0 \frac{\partial \Phi}{\partial t} + p + \tfrac{1}{2}\rho_0 \frac{\gamma^2}{r^2} + \rho_0 \frac{\gamma}{r^2}\frac{\partial \Phi}{\partial \phi} = p_0. \tag{48}$$

In (48), Φ represents the superposed potential due to the direct and scattered fields

$$\Phi = \int_{-\infty}^{\infty} \sum_{-\infty}^{\infty} [B_m H_m(\beta r) + Q J_m(\beta r) H_m(\beta r_0)] \times \exp[im(\phi - \phi_0)] \exp(i\alpha z) \exp(-i\omega t)\, d\alpha \quad (r < r_0). \tag{49}$$

The requirement, that the material derivative of the pressure should vanish, gives a unique value for B_m, the amplitude of each mode of the scattered field:

$$B_m = -Q\left[\frac{[\omega - (\gamma m/a^2)]^2 H_m(\beta r_0) - (\gamma^2/a^3)\beta H'_m(\beta r_0)}{[\omega - (\gamma m/a^2)]^2 H_m(\beta a) + (\gamma^2/a^3)\beta H'_m(\beta a)}\right] J_m(\beta a) \quad (m \neq 0), \tag{50}$$

$$B_0 = -Q\left[\frac{\omega^2 H_0(\beta r_0) - (\gamma^2/a^3)\beta H'_0(\beta r_0)}{\omega^2 H_0(\beta a) + (\gamma^2/a^3)\beta H'_0(\beta a)}\right] J_0(\beta a). \tag{51}$$

When the distance of the source from the cavity centre is of the same order as the radius of the cavity, i.e. $r_0 \sim a$, then, for $\beta a \ll 1$, this becomes

$$B_m = -Q\left[\frac{[(\omega/\Omega) - m]^2 + |m|}{[(\omega/\Omega) - m]^2 - |m|}\right] J_m(\beta a) \quad (m \neq 0), \tag{52}$$

$$B_0 = -Q\left[\frac{(\omega/\Omega)^2 \ln \beta a - 1}{(\omega/\Omega)^2 \ln \beta a + 1}\right] J_0(\beta a). \tag{53}$$

It follows from (52) that a resonance is possible for modes corresponding to non-zero integers at a frequency given by

$$\omega_m = (m \pm |m|^{\frac{1}{2}})\,\Omega. \tag{54}$$

Since the resonant frequency ω_m, given by (54), changes sign with m, it follows that the angular phase velocity of the resonant perturbation field is always with the mean flow. No resonance of the volume pulsation mode, $m = 0$, can occur, a point already clear from (21). It is seen from (54) that at the resonant condition, the angular phase velocity exceeds that of the mean flow, and increases with mode order. These resonances are precisely those found by Kelvin for a hollow vortex in incompressible flow. Our expectation that the acoustic energy loss would stabilize those modes was not realized, so that the resonant modes emerge as the dominant feature of the radiation field. We shall return to this point later to show how these resonances can be driven to high amplitude where the flow model must fail.

The contribution to the scattered field of modes corresponding to $m = 1, 2, 3, \ldots$ is

$$I = \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} B_m H_m(\beta r) \exp[im(\phi - \phi_0)] \exp(i\alpha z) \exp(-i\omega t)\, d\alpha, \tag{55}$$

where B_m has the value given in (52). If (ω/Ω) has none of the resonant values, then B_m, the amplitude of each scattered mode, is bounded for all m, and, for $\beta a \ll 1$, (55) can be shown to be of order

$$I \sim \left(\frac{\exp(ikr)}{r}\right) O(ka) \tag{56}$$

by stationary phase integration, for large values of kr.

It can be seen from (53) that the radiation from the volume pulsation mode, $m = 0$, is like $Q\exp(ikr)/r$ for large kr, so that the scattered and direct fields are of the same order, the scattered field being dominated by the volume pulsation mode. The contribution from all other modes can be seen to be an order of magnitude less. If the parameter $(\omega/\Omega)^2|\ln ka|$ has value unity, then there is no radiation from the volume pulsation mode, as follows from (53).

If surfaces are present in the flow, as in the case of an underwater propeller, then the incident field is dipole induced, and may be given by

$$\Phi = -D_r \frac{\partial}{\partial r}\left(\frac{\exp(ikR)}{R}\right). \tag{57}$$

The scattered field can be computed in exactly the manner described above, and the amplitude of each scattered mode is found to be ($r_0 \sim a$, $\beta a \ll 1$)

$$B_m = \frac{|m| D_r}{a}\left[\frac{(\omega/\Omega - m)^2 + (|m|+1)}{(\omega/\Omega - m)^2 - |m|}\right] J_m(\beta a) \quad (m \neq 0), \tag{58}$$

$$B_0 = -\frac{D_r}{a}\left[\frac{(\omega/\Omega)^2 + 1}{(\omega/\Omega)^2 \ln \beta a + 1}\right] J_0(\beta a). \tag{59}$$

As before, the contribution from modes corresponding to non-zero integers can be shown to be at least an order of magnitude less than that of the volume pulsation mode $m = 0$, which gives the following long wavelength comparison of scattered, Φ_S, and direct, Φ_D, fields ($kr \to \infty$)

$$\left|\frac{\Phi_S}{\Phi_D}\right| \sim \frac{1}{ka} \qquad \left[\left(\frac{\omega}{\Omega}\right)^2 \ln ka \ll 1\right], \tag{60}$$

$$\left|\frac{\Phi_S}{\Phi_D}\right| \sim \frac{1}{ka}|\ln ka| \quad \left[\left(\frac{\omega}{\Omega}\right)^2 \ln ka \gg 1\right]. \tag{61}$$

If the dipole axis is tangential to the vortex flow, the potential field is given by

$$\Phi = D_\phi \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{\exp(ikR)}{R}\right). \tag{62}$$

Then if, $r_0 \sim a$, $\beta a \ll 1$, the amplitude of each scattered mode is calculated to be

$$B_m = -\frac{im}{a} D_\phi \left[\frac{(\omega/\Omega - m)^2 + |m|}{(\omega/\Omega - m)^2 - |m|}\right] J_m(\beta a) \quad (m \neq 0), \tag{63}$$

$$B_0 = 0. \tag{64}$$

It is seen that the scattered and direct fields are of equal order for large values of kr, with no radiation from the volume pulsation mode.

When the incident field of an axial dipole, i.e.

$$\Phi = D_z \frac{\partial}{\partial z}\left(\frac{\exp(ikR)}{R}\right) \tag{65}$$

is considered, the amplitude of each scattered mode is found to be ($\beta a \ll 1$, $r_0 \sim a$)

$$B_m = -i\alpha D_z \left[\frac{(\omega/\Omega - m)^2 + |m|}{(\omega/\Omega - m)^2 - |m|}\right] J_m(\beta a) \quad (m \neq 0), \tag{66}$$

$$B_0 = -i\alpha D_z \left[\frac{(\omega/\Omega)^2 \ln(\beta a) - 1}{(\omega/\Omega)^2 \ln(\beta a) + 1}\right] J_0(\beta a). \tag{67}$$

When the scattered field is evaluated by an integral analogous to (55), the stationary point of the integral is found to be at

$$\alpha = k\cos\theta \tag{68}$$

for large values of r and z, θ being the polar angle at the observation point. It is seen that, provided the parameter $(\omega/\Omega)^2 |\ln(ka\sin|\theta|)|$ is not of order unity, the symmetric volume pulsation mode scatters to infinity a field of strength equal to the incident field. When $(\omega/\Omega)^2 |\ln(ka\sin|\theta|)|$ is of order unity, the scattered field vanishes to within $O(ka)$.

Suppose now that the vortex is driven by a region of turbulence. In this case, the incident field would be quadrupole with typical potential

$$\Phi = T_{rr}\frac{\partial^2}{\partial r^2}\left(\frac{\exp(ikR)}{R}\right). \tag{69}$$

The amplitude of each scattered mode is calculated to be

$$B_m = -\frac{T_{rr}}{a^2}|m|(|m|+1)\left[\frac{(\omega/\Omega - m)^2 + (|m|+2)}{(\omega/\Omega - m)^2 - |m|}\right] J_m(\beta a) \quad (m \neq 0), \tag{70}$$

$$B_0 = \frac{T_{rr}}{a^2}\left[\frac{(\omega/\Omega)^2 \ln\beta a + 2}{(\omega/\Omega)^2 \ln\beta a + 1}\right] J_0(\beta a). \tag{71}$$

The volume pulsation mode $m = 0$ dominates the scattered field, giving the following ratio of scattered, Φ_S, to direct, Φ_D, fields ($kr \to \infty$):

$$\left|\frac{\Phi_S}{\Phi_D}\right| \sim \left(\frac{1}{ka}\right)^2 \qquad \{(\omega/\Omega)^2 \ln ka \ll 1\} \tag{72}$$

$$\left|\frac{\Phi_S}{\Phi_D}\right| \sim \frac{1}{(ka)^2|\ln ka|} \quad \{(\omega/\Omega)^2 \ln ka \gg 1\}. \tag{73}$$

When the axes of the quadrupole are along the radius and the z-axis respectively, i.e.

$$\Phi = T_{rz}\frac{\partial^2}{\partial r\,\partial z}\left(\frac{\exp(ikR)}{R}\right),$$

the amplitude of each scattered mode is found as follows: ($\beta a \ll 1, r_0 \sim a$)

$$B_m = -i\alpha\frac{|m|}{a} T_{rz}\left[\frac{(\omega/\Omega - m)^2 + (|m|+1)}{(\omega/\Omega - m)^2 - |m|}\right] J_m(\beta a) \quad (m \neq 0), \tag{74}$$

$$B_0 = -\frac{i\alpha}{a} T_{rz}\left[\frac{(\omega/\Omega)^2 + 1}{(\omega/\Omega)^2 \ln\beta a + 1}\right] J_0(\beta a). \tag{75}$$

Analogous to the case of an axial dipole, it follows that, when

$$(\omega/\Omega)^2\,|\ln(ka\sin|\theta|) \ll 1,$$

the scattered field strength is of order $(1/ka)$ times the direct field, while, for the other limit, $(\omega/\Omega)^2\,|\ln(ka\sin|\theta|)| \gg 1$, the scattered field strength is of order $1/ka\,|\ln ka|$ times the direct field.

When the incident field is due to the quadrupole

$$\Phi = T_{z\phi}\frac{1}{r}\frac{\partial^2}{\partial z\,\partial\phi}\left(\frac{\exp(ikR)}{R}\right), \tag{76}$$

it is found that the scattered field has strength equal to the direct field, with no radiation from the volume pulsation mode.

When the incident potential is given as

$$\Phi = -T_{r\phi}\frac{1}{r}\frac{\partial^2}{\partial\phi\,\partial r}\left(\frac{\exp(ikR)}{R}\right), \tag{77}$$

the scattered mode amplitude for each m is found to be

$$B_m = \frac{im^2}{a^2}T_{r\phi}\left[\frac{(\omega/\Omega-m)^2+(|m|+1)}{(\omega/\Omega-m)^2-|m|}\right]J_m(\beta a) \quad (m \neq 0), \tag{78}$$

$$B_0 = 0. \tag{79}$$

It follows that the scattered field is of order $(1/ka)$ times the direct field strength, with no radiation from the symmetric mode, $m = 0$. It is easy to see that the same result applies to the quadrupole field given by

$$\Phi = T_{\phi\phi}\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\left(\frac{\exp(ikR)}{R}\right). \tag{80}$$

As for quadrupoles involving an axial direction, it can be shown in a manner similar to the above that, for the $(r-z)$ quadrupole,

$$\left|\frac{\Phi_S}{\Phi_D}\right| \sim \left(\frac{1}{ka}\right) \qquad \left[\left(\frac{\omega}{\Omega}\right)^2\ln(ka\sin\theta) \ll 1\right] \tag{81}$$

$$\left|\frac{\Phi_S}{\Phi_D}\right| \sim \frac{1}{ka\,|\ln ka|} \quad \left[\left(\frac{\omega}{\Omega}\right)^2\ln(ka\sin\theta) \gg 1\right], \tag{82}$$

where Φ_S and Φ_D denote the scattered and direct potentials, respectively.

Finally, the $(z-z)$ quadrupole has both the direct and the scattered fields of equal strength, provided that $(\omega/\Omega)^2\,|\ln(ks\sin|\theta|)| \neq 1$, for which value there is no scatter from the symmetric mode $m = 0$.

An initial value problem

In this section we consider the initial value problem, where the vortex is disturbed from rest by a point impulse applied at time $t = \tau$.

The solution to this problem is obtained from the periodic source solution by integrating over all possible frequencies, which by (52) and (53) gives Φ as a generalized function:

$$\Phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{\substack{-\infty\\ m\neq 0}}^{\infty}\left[\frac{[(\omega/\Omega)-m]^2+|m|}{[(\omega/\Omega)-m]^2-|m|}\right] J_m(\beta a)\,H_m(\beta r)$$
$$\times \exp[im(l-l_0)]\exp(i\alpha z)\exp[-i\omega(t-\tau)]\,d\alpha\,d\omega\,H(t-\tau)$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{(\omega/\Omega)^2\ln\beta a-1}{(\omega/\Omega)^2\ln\beta a+1}\right] J_0(\beta a)\,H_0(\beta r)$$
$$\times\exp(i\alpha z)\exp[-i\omega(t-\tau)]\,H(t-\tau)\,d\alpha\,d\omega. \tag{83}$$

The causality condition has been used to eliminate the free-wave solutions. The second term in (83) gives no contribution, since $[(\omega/\Omega)^2\ln\beta a+1]$ never vanishes. Performing the α integration in the first term gives

$$\Phi = \int_{-\infty}^{\infty}\sum_{\substack{-\infty\\ m\neq 0}}^{\infty}\left[\frac{[(\omega/\Omega)-m]^2+|m|}{[(\omega/\Omega)-m]^2-|m|}\right] J_m(ka)$$
$$\times\exp(ikr)\exp[\tfrac{1}{2}i\pi(m+1)]\exp[-i\omega(t-\tau)]\,H(t-\tau)\,d\omega, \tag{84}$$

which can be evaluated as

$$\Phi = \pi i\sum_{\substack{-\infty\\ m\neq 0}}^{\infty}|m|^{\frac{1}{2}}\exp[\tfrac{1}{2}i\pi(m+1)]\left[J_m(k_1a)\frac{\exp(ik_1r)}{r}\exp[-i\omega_1(t-\tau)]\right.$$
$$\left. -J_m(k_2a)\frac{\exp(ik_2r)}{r}\exp[-i\omega_2(t-\tau)]\right]H(t-\tau), \tag{85}$$

where

$$\left.\begin{aligned}\omega_1 &= (m+|m|^{\frac{1}{2}})\,\Omega,\\ \omega_2 &= (m-|m|^{\frac{1}{2}})\,\Omega,\end{aligned}\right\} \tag{86}$$

and k_1 and k_2 are the associated wave-numbers. It is seen from (86) that the field of a point source applied at time $t=\tau$, satisfying the required boundary conditions, is a superposition of the resonant modes. These survive indefinitely, even though energy is lost to their associated acoustic field.

To derive the response of the vortex to a periodic excitation at a resonant frequency ω_1 switched on at a time $t=0$, we form the convolution of

$$H(t)\exp(-i\omega_1 t),$$

with (85), i.e. the response from a point impulse at time $t=\tau$. This gives

$$\Phi = \pi i\int_{-\infty}^{\infty}H(\tau)\exp(-i\omega_1\tau)\sum_{\substack{m\neq 0\\ -\infty}}^{\infty}\exp[\tfrac{1}{2}i\pi(m+1)]\left[J_m(k_1a)\frac{\exp(ik_1r)}{r}t\right.$$
$$\left.\times\exp[-i\omega_1(t-\tau)]-J_m(k_2a)\frac{\exp(ik_2r)}{r}\exp[-i\omega_2(t-\tau)]\right]H(t-\tau)\,d\tau \tag{87}$$
$$= \pi i\sum_{\substack{m\neq 0\\ -\infty}}^{\infty}|m|^{\frac{1}{2}}\exp[\tfrac{1}{2}i\pi(m+1)]\left[J_m(k_1a)\frac{\exp(ik_1r)}{r}t\exp(-i\omega_1t)\right.$$
$$+iJ_m(k_2a)\frac{\exp(ik_2r)}{r}\frac{\exp(-i\omega_1t)}{\omega_1-\omega_2}$$
$$\left.-iJ_m(k_2a)\frac{\exp(ik_2r)}{r}\frac{\exp(-i\omega_2t)}{\omega_1-\omega_2}\right]H(t). \tag{88}$$

From this expression, it is seen that, when the vortex is excited by a periodic disturbance at the resonant frequency ω_1 switched on at time $t = 0$, the response is a superposition of the resonant modes of ω_1 and ω_2, the amplitude of the ω_1 modes increasing without bound. This algebraic instability will eventually lead to breakdown of the flow model.

Summary and conclusions

The scattering properties of a multipole driven cavitated line vortex have been considered. We find that the dominant feature is that Kelvin's resonant modes remain undamped by compressibility effects, and can exist in a neutral condition, despite the continual energy demands of the distant radiation field. When excited by an external source those modes grow without limit according to linear theory, so that the flow is subject to a driven instability. The off-resonance scattered field is a function of the parameter $(\omega/\Omega)^2 |\ln ka|$, which measures the relative importance of the compressibility and inertia effects in the exterior fluid, compared to the 'stiffness' of the cavity arising from the radial pressure gradient in the steady vortex flow.

The near field of a line source is scattered to infinity, its strength there being of order $S^2 \ln ka$ times the direct field when $(\omega/\Omega)^2 |\ln ka| \ll 1$. On the other hand, when $(\omega/\Omega)^2 |\ln ka| \gg 1$, the scattered and direct fields are of equal strength at infinity. The near field of a radial line dipole is scattered to infinity with strength of order S^2/ka or $1/(ka |\ln ka|)$ times the direct field, according as $(\omega/\Omega)^2 |\ln ka|$ is small or large compared to unity, respectively, while, for a radial line quadrupole, its near field is scattered with strength of order $(S/ka)^2$ or $(1/ka)^2 \, 1/|\ln ka|$, according as the parameter $(\omega/\Omega)^2 |\ln ka|$ is small or large.

It is found that the near field of a simple point source is scattered with strength of the same order as that of the direct field, provided $(\omega/\Omega)^2 |\ln ka| \neq 1$. When this parameter has value unity, there is no radiation from the volume pulsation mode, $m = 0$. At all other conditions, the scattered field is dominated by the radiation from the zeroth mode, the contribution of all other modes being an order of magnitude less.

The near field of a radial point dipole is scattered with strength proportional to $1/ka$ or $1/ka |\ln ka|$ times the direct field strength, according as $(\omega/\Omega)^2 |\ln ka|$ is small or large compared to unity, respectively, while the near field of a radial point quadrupole is scattered with strength proportional to $(1/ka)^2$ or $(1/ka)^2$ $1/|\ln ka|$. The scattered fields are again dominated by radiation from the volume pulsation mode, $m = 0$.

In the case of a tangential dipole, its near field is scattered with strength of the same order as the direct field, no radiation arising from the zeroth mode. Both the scattered and direct fields of an axial dipole are also of equal strength, provided $(\omega/\Omega)^2 |\ln (ka \sin |\theta|)| \neq 1$. When this condition is violated, the radiation from the volume pulsation mode vanishes.

The fields induced by quadrupoles other than the radial quadrupole, and quadrupoles involving an axial direction, have strengths $O(1/ka)$ times the direct fields at infinity. For the quadrupole, which has one axis along the z-axis, and the other along the radius, the scattered field has strength $O(1/ka)$ or $O(1/ka\,|\ln ka|)$ times the direct field, depending on the magnitude of the parameter

$$(\omega/\Omega)^2\,|\ln(ka\sin|\theta|)|.$$

The scattered fields of both the $(\phi-z)$ quadrupole and $(z-z)$ quadrupole are equal in strength at infinity, except when $(\omega/\Omega)^2\,|\ln(ka\sin|\theta|)| = 1$ in the case of the $(z-z)$ quadrupole. At this value of the parameter, the radiation from the volume pulsation mode, which is otherwise the dominant mode, vanishes.

On linear theory, there is infinite scattering at the vortex resonance frequencies, a condition that will no doubt lead to breakdown of the equations—and possibly the vortex!

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